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The Weakly Neighborly Polyhedral Maps on the Non-orientable 2-Manifold with Euler Characteristic -2

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A weakly neighborly polyhedral map (w.n.p. map) is a 2-dimensional cell-complex which decomposes a closed 2-manifold without boundary, such that for every two vertices there is a 2-cell containing them. We prove that there are just eight non-orientable w.n.p. maps with Euler characteristic -2 and we describe them.

1. INTRODUCTION

A *weakly neighborly polyhedral map* (w.n.p. map) is a 2-dimensional topological cell-complex which decomposes a closed 2-manifold (usually without boundary), such that for every two vertices there is a 2-cell containing them. The 0-, 1-, and 2-dimensional cells of the map are its *vertices*, *edges*, and *facets*, respectively.

Some aspects of w.n.p. maps of arbitrary genus have been studied in [BA1], where also a detailed study of w.n.p. maps on the sphere is given. (The genus of a 2-manifold is defined as $g = \frac{1}{2}(2-\chi)$, where χ denotes the Euler characteristic of the manifold. Thus the projective plane is of genus $\frac{1}{2}$, and the genus of the 2-manifold studied in the present paper is 2.) In [BA2] a complete list of the 5 w.n.p. maps on the torus is given. In [AB1] we prove that there are no orientable w.n.p. maps of genus 2. Complete lists of the non-orientable w.n.p. maps with Euler characteristics 1, 0, and -1 are given in [B, AB2].

In the present paper we investigate the non-orientable w.n.p. maps with Euler characteristic -2 (i.e., $g = 2$). We prove

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THEOREM. *There are precisely eight non-orientable* w.n.p. *maps with Euler characteristic* -2. *They are the maps depicted in Figs.* 1, 2.

The proof of our theorem is heavily based on some of the results obtained in [BA1, AB1]. We refer the reader to [BA1] also for motivation for the concept of a w.n.p. map. There we prove that the number of w.n.p. maps on any 2-manifold N other than the 2-sphere is finite, and we give an upper bound for the number of vertices in a w.n.p. map on N, as a function

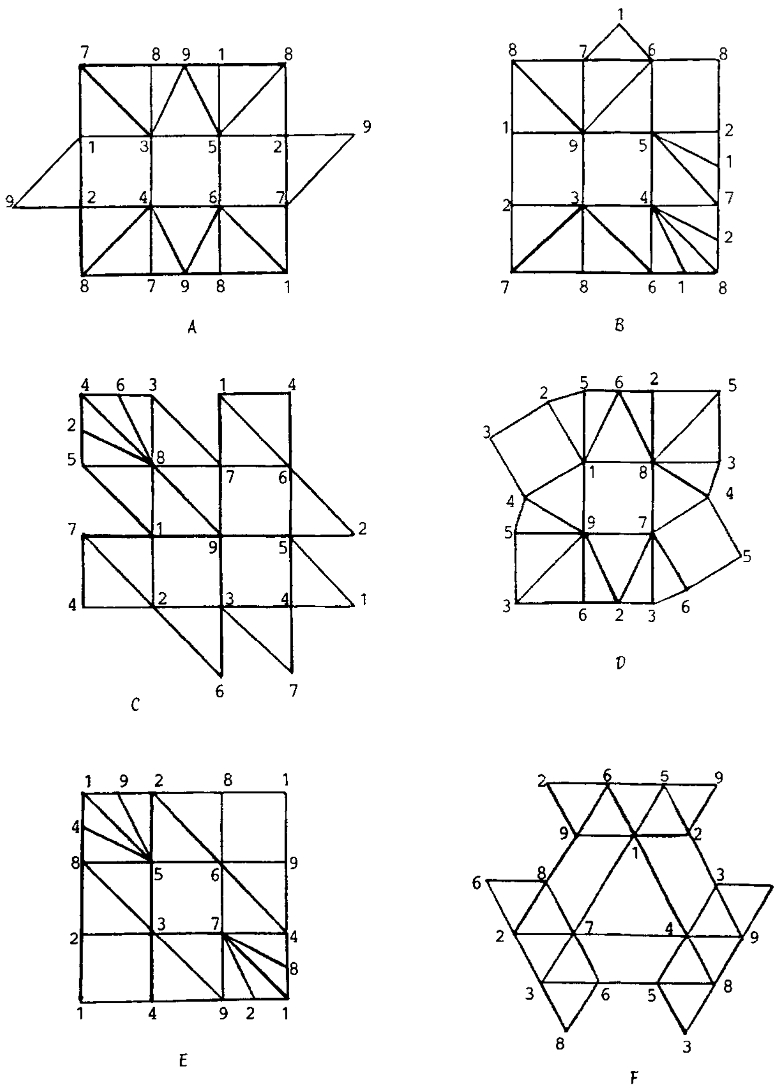


FIG. 1. The six w.n.p. maps with $V = 9$, $p = (16, 3, 0, \ldots)$.

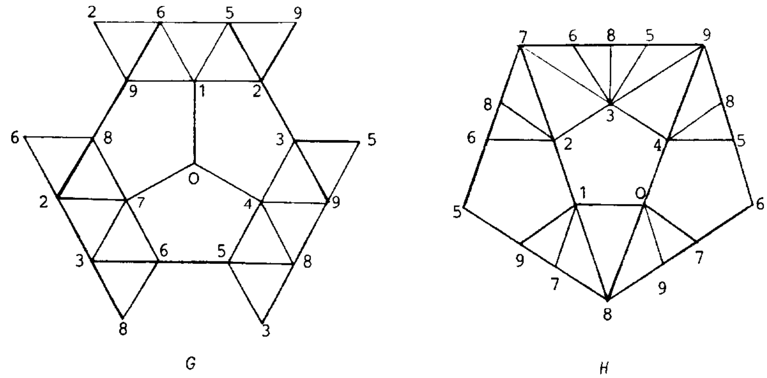


FIG. 2. The two w.n.p. maps with $V = 10, p = (15, 0, 3, 0, \ldots)$.

of the genus g of N. For $g = 2$ this upper bound is 12. As a large part of the proof of our theorem, and also of the proof of the main theorem in [AB1], relies merely on the fact that the genus of the maps dealt with in those two theorems is 2, and not on the orientability of the maps, we may— and do— refer often to [AB1].

The following will serve as our standard notation. We assume a polyhedral (or w.n.p.) map M of genus $g = 2$ with V vertices $v_1, \ldots, v_V$ and F facets $f_1, \ldots, f_F$. k_i is the number of vertices of f_i $(1 \leqslant i \leqslant F)$, and we take the notation to be such that $k_1 \geqslant k_2 \geqslant \cdots \geqslant k_F$. The vertices, too, are labelled according to decreasing order of valences (degrees), that is, $\deg v_1 \geqslant \deg v_2 \geqslant \cdots \geqslant \deg v_V$. p_i denotes the number of i-gonal facets of the map M, and $p = (p_3, p_4, \ldots)$ is the p-vector of M. V_i denotes the number of vertices of valence i, and $(V_3, V_4, \ldots)$ is the *v-vector* of M.

The proof of our theorem consists of two parts. First, in Section 2, we find a set of candidates for p-vectors such that every non-orientable w.n.p. map with $g = 2$ has its p-vector in the set. The second part (Sects. 3, 4) consists of a detailed investigation of each of those candidates. In each of those cases we assume that there is some w.n.p. map realizing that "p-vector." In some of the cases this leads to a contradiction (Sect. 3), and in the remaining cases we find all the possible maps (Sect. 4). Finally, in Section 5, we describe the symmetry groups of the eight maps mentioned in our theorem; we describe how four of those maps can be obtained from certain w.n.p. maps of lower genus by using certain transformations, and we describe a construction for increasing or decreasing the number of vertices in a w.n.p. map.

For the reader's convenience, we quote some of the results proved in [BA1] which will be used here, with their reference numbers. Those results are adapted here for $g = 2$. They are as follows. ($\lceil x \rceil$ denotes the least integer $\geqslant x$, and $|f_i \cap f_j|$ denotes the number of vertices common to the facets f_i, f_j.) The following hold for every polyhedral map with $g=2$:

$$2V+4=\sum_{i=1}^{F}(k_i-2) \tag{1}$$

$$k_i \geqslant 3 \qquad \text{for every} \quad 1 \leqslant i \leqslant F, \tag{2}$$

The following hold for every w.n.p. map with $g=2$:

$$V^2-7V-12=\sum_{i=1}^{F}(k_i-2)(k_i-3). \tag{6}$$

$$k_1 \geqslant \left[\frac{V}{2}\right]-1 \tag{7}$$

$$\sum_{|f_i \cap f_j|=2} k_i = \sum_{f_i \cap f_j=\varnothing}(k_i-2)=(k_j-2)(V-k_j+2)-2$$
$$\text{for every} \quad 1 \leqslant j \leqslant F, \tag{10}$$

$$\sum_{i=2}^{k_1+1} k_i \geqslant (k_1-2)(V-k_1+2)-2. \tag{13}$$

$$k_1 \leqslant \max\{6, \tfrac{1}{2}(V+1)\}. \tag{20}$$

From [BA2] we quote:

LEMMA 1. *If v is a vertex of valence l in* w.n.p. *map M and the l facets of M incident to v have $k_{i_1}, k_{i_2}, \ldots, k_{i_l}$ vertices, then $\sum_{j=1}^{l} k_{i_j} = V+2l-1$.*

LEMMA 2. *If M is a polyhedral map and f is a facet of M with l vertices $v_{i_1}, v_{i_2}, \ldots, v_{i_l}$, then $\sum_{j=1}^{l} \deg v_{i_j} \leqslant F+2l-1$.*

2. Outline of the Proof

In [AB1] we investigated the *orientable* w.n.p. maps of genus 2. In large parts of that investigation we did not use the orientability, and therefore those parts are applicable here too. In particular, in finding the candidates for p-vectors we used the orientability only to exclude the case $V<10$. For $V \geqslant 10$ we found eleven candidates for p-vectors. All these candidates were shown to be non-realizable, and in seven of them, namely those with $V>10$, we did not use the orientability in the process of the proof. Thus it remains to investigate the four candidates for p-vectors with $V=10$ (they are listed in Table I), and to investigate the case $V<10$.

TABLE I

The Candidates for p-Vectors of the Non-orientable Maps of Genus 2

Case	V	p_3	p_4	p_5	No. of realizations
1	9	16	3	0	6
2	10	15	0	3	2
3	10	12	3	2	0
4	10	9	6	1	0
5	10	6	9	0	0

An elementary consideration yields that $V \geqslant 9$ for every 2-complex with V vertices whose body is a 2-manifold of genus 2 (orientable or not). For $V=9$ it is easily seen that there are just two candidates for p-vectors which are not in direct contradiction to (1), (2), (6), (7), and (20), namely $(19, 0, 1, 0, \ldots)$ and $(16, 3, 0, \ldots)$. The first of these violates (13), and we are left with just one candidate. We shall see that this candidate, which does not exist at all in the orientable case, yields six distinct w.n.p. maps!

Altogether we have five candidates for p-vectors, and they are listed in Table I, where we also show the number of distinct topological realizations for each of these p-vectors, as proved in the next two sections. In all the five cases, $p_i = 0$ for $i \geqslant 6$.

In the process of examining a "p-vector" p we often draw the w.n.p. map M which is supposed to realize p, or a fragment of it, M is drawn as a planar map with identification on the boundary edges and vertices. As each vertex must "see" all the other $V-1$ vertices, the $V-1$ vertices which see a certain vertex must be distinct. We use the labels $1, \ldots, V$ (0 stands for 10) to indicate such a set of V distinct vertices, and the labels $a, b, \ldots$ to indicate vertices that must be identified with some vertices in the set $\{1, \ldots, V\}$. A broken line-segment indicates a line segment which either exists, that is, as an edge in M, or does not exist, that is, it is a diagonal of some facet in M.

We also use the following notation. A_{abc} means that it happens twice that b is adjacent to both vertices a and c on the boundary of the configuration, and hence it is already completely surrounded by facets (see [AB1, Fig. 3c])—usually too few of them. B_{ab} means that the edge ab belongs to two facets (and therefore, if on the boundary of the configuration, it must appear twice). C_a denotes that the facets which contain the vertex a either do not close to a disc, or they form a disc which is not the correct one. In particular (mostly in the discussion of Case 5), we write C_a^- if the disc is too small, and C_a^+ if it is too large. (Note that if the vertex a occurs m times on the boundary of a planar representation of a w.n.p. map with V vertices, then the links of these m occurrences of a should contain together precisely $V+m-1$ seemingly different vertices.) D_{ab} indicates that ab is a diagonal in some facet (and usually an edge or a diagonal in another facet too, which means a contradiction) and E_{ab} means simply that ab is (or must be) an edge.

3. Three Non-realizable Cases

In this section we show that each of Cases 3, 4, 5 of Table I has no topological realization.

Case 3. Here $V = 10$, $p = (12, 3, 2, 0, \ldots)$. This case has been dealt with in [AB1] and, with the additional assumption of orientability, was shown to be non-realizable. However, orientability has been assumed only in one subcase—namely the case of Fig. 4f in [AB1]—so that here we have to investigate just that case.

This case is depicted here again, in Fig. 3a, where precisely one of the dotted edges is not an edge. As all the facets here touch one of the pentagons and are therefore distinct, the entire map is depicted here, provided

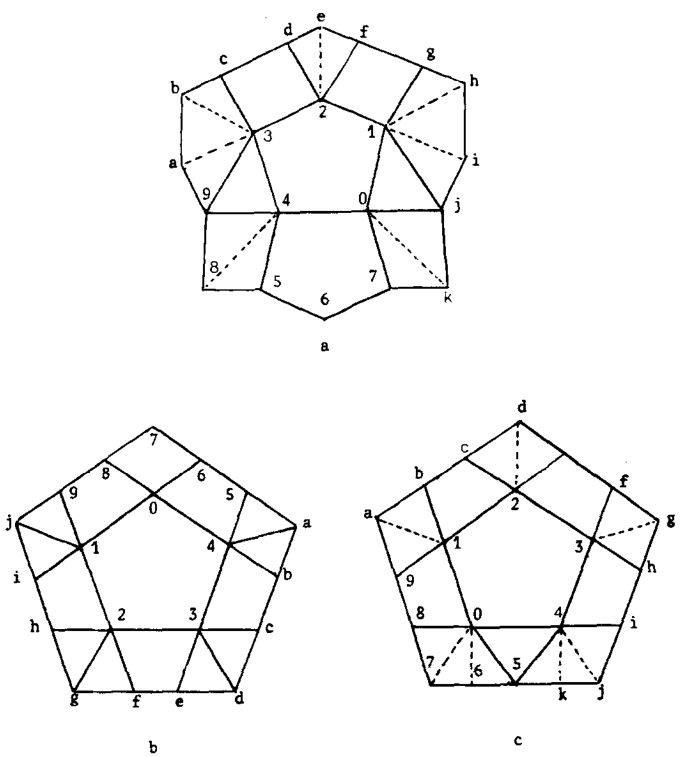


FIGURE 3

we can identify the vertices $a, b, \ldots, k$ with $0, 1, \ldots, 9$ in a proper way. Now we will show that this is not possible.

Assume, on the contrary, that this is possible. Then $\{j, k\} = \{8, 9\}$. Consider the possibility $k=8$. Then $j=9$; 9 sees 1 and 3 and must also see 2, hence $e=9$. There are just three occurencies of 9 on the boundary of the configuration, hence $\{d, f\} = \{a, i\}$ and, as $a \neq d$ (both see 3) we have $a=f$, $d=i$. $B_{89} \Rightarrow 8 \notin \{d, f\} \Rightarrow 8 \in \{g, h\}$ and $8 \in \{b, c\}$. As 2 must see 8, we have also $8 \in \{c, g\}$. Thus, because of the symmetry of the configuration, we may take $g=b=8$. Since $a=f$, we get the contradiction A_{8a9}. Therefore $k \neq 8$.

Thus $k=9$, $j=8$; $i \neq 5$ (otherwise A_{589}), hence $i \in \{6, 7\}$. Similarly $a \in \{5, 6\}$.

Assume $a=5$. Then $D_{57} \Rightarrow b \neq 7 \Rightarrow b \in \{6, 8\}$. If $b=8$ then the only possibility for 2 to see 8 is $e=8$. Clearly $\{c, d\} = \{6, 7\}$ which implies $f, g \notin \{6, 7\}$, and this in turn implies $\{h, i\} = \{6, 7\}$. Thus the edge 67 appears three times—a contradiction. Hence $b=6$; $c \neq 7$ (otherwise A_{567}) implies $c=8$, $d=7$. Now B_{58} implies $i=5$, and we have the contradiction A_{589}.

Therefore $a=6$; $D_{57} \Rightarrow c=8$. There are just these three occurrences of 8 on the boundary, hence $\{b, d\} = \{5, 7\}$ implies $i=7$; $b \neq 5$ (otherwise A_{658}) $\Rightarrow b=7$, $d=5$; $\{e, f, g\} = \{6, 7, 9\}$ and $i=7$ imply $e=7$, and 57 is now both an edge and a diagonal—a contradiction. Hence Case 3 is not realizable.

Case 4. Here $V=10$, $p=(9, 6, 1, 0, \ldots)$. Assume that there is such a w.n.p. map M. Lemma 1 implies that $V_3 = 0$ and that the pentagon and three quadrangles meet at each 4-valent vertex. (10) (with $j=1$) implies that the pentagon has a common edge with at least four quadrangles. Thus we distinguish two cases.

(a) The pentagon is adjacent to five quadrangles (Fig. 3b). Note that exactly one triangle of the map M is missing in Fig. 3b. If $b=8$ then $\{a, c\} = \{7, 9\}$ and we get the contradiction A_{789}. Hence $b \neq 8$, and similarly $i \neq 6$. Thus $b \in \{7, 9\}$ and $i \in \{5, 7\}$.

Assume $b=7$. Then $\{a, c\} = \{8, 9\}$.

If $a=8$ then $c=9$; 9 must appear again, and the only possibility for this is $g=9$. As $\{d, e, f\} = \{5, 6, 8\}$, D_{68} implies $e=5$ and this, in turn, implies $h, i \neq 5$ (as e, h, and i see 2). Thus $i=7$, $h=6$, $j=5$. As 2 and 3 must see all the vertices we must have $f=8$, $d=6$. Now each of the edges 57, 59, 79 appears just once on the boundary of the configuration. Thus the missing triangle must be 579, and this leads to the contradiction C_5.

Hence $a=9$, $c=8$; 8 must appear again, hence $g=8$. $D_{68} \Rightarrow 6 \notin \{d, f, h\} \Rightarrow e=j=6$; $d \neq 9$ (otherwise A_{789}) implies $d=5$, $f=9$, and we get the contradiction A_{698}.

Thus $b \neq 7$, and because of the symmetry also $i \neq 7$. Hence $b=9$, $i=5$. The only way for 9 to appear again is $g=9$. Since E_{89} and $8 \in \{a, c\}$, we get $8 \notin \{f, h\}$. Thus, as 2 must see 8, we have $e=8$. The symmetry of the configuration now implies $f=6$, which is impossible as 68 is now both an edge and a diagonal.

(b) The pentagon is adjacent to four quadrangles and a triangle. Figure 3c depicts the entire map, and precisely two of the dotted edges are diagonals. As $\{a, b, c\} = \{5, 6, 7\}$, we have $b \neq 6$ (otherwise A_{567}). Thus one of the edges ab or bc is 57, and therefore 06 is an edge. From the symmetry of the map it follows that $k \neq f$ and $4k$ is an edge.

Assume $b=5$. Then $D_{25} \Rightarrow 5 \notin \{e, f\}$ and clearly $5 \notin \{i, h\}$. Hence $g=5$. In order to avoid the contradiction C_5, $1a$ must be a diagonal and $\{f, h\} = \{7, k\}$ (because $\{a, c\} = \{6, 7\}$). Since $k \neq f$ we have $k=h$, but k and h both see 4—a contradiction.

Thus $b=7$, and by the symmetry also $f=j$. If $a=6$ then $c=5$ and this implies, as before, that $g=5$. As 3 must see 7, 2 already sees 7, and the edge 57 must appear twice, we have $h=7$. It now follows easily (comparing the link of 3 with those of 2 and 4) that $i=d$ and $k=e$. $B_{78} \Rightarrow i=8$. Now $\{e, f\} = \{j, k\} = \{6, 9\}$ and the edge 69 appears three times. Thus $a \neq 6$, which yields $a=5$, $c=6$. As 3 must see the vertices 6, 7, the vertex 2 already sees them and the edge 67 already appears twice, we get $\{g, i\} = \{6, 7\}$, which implies $d=h \neq 5$ (as 4 already sees 5). As 3 must see 5 and $f=j \neq 5$, we get $e=5$. Now, in order to avoid the contradiction C_5^-, $2d$ must be a diagonal, but then 56 is both an edge and a diagonal—a contradiction. Thus we conclude that Case 4 has no topological realization.

Case 5. Here $V=10$, $p=(6, 9, 0,...)$. This case has been dealt with in [AB1] and, with the additional assumption of orientability, was shown to be non-realizable. However, orientability has been assumed there for the first time in Step 3, hence Steps 1, 2 are valid here too. Thus we conclude that if our case is realizable by some map M, then $V_5=6$, $V_6=4$, four quadrangles and one triangle meet at each 5-valent vertex, three quadrangles and three triangles meet at each 6-valent vertex, the configuration of Fig. 4a is a part of the map M, and all the facets of M intersect the quadrangle $wxyz$.

As the vertex y belongs to two triangles, y must be of valence 6. deg w is either 5 or 6, and we obtain a contradiction by showing that none of these is possible.

Assume deg $w=6$. Lemma 2 implies that deg $x=$ deg $z=5$, so we obtain the configuration of Fig. 4b (where 1230 is the quadrangle $wxyz$). This configuration depicts the entire map M, and all the facets here are distinct, as they all touch the quadrangle 0123. Precisely one of $3i$, $3j$ is a diagonal.

If $d=7$, then $\{a, b, c\} = \{4, 5, 6\}$. Now if $b=5$ we get the contradiction

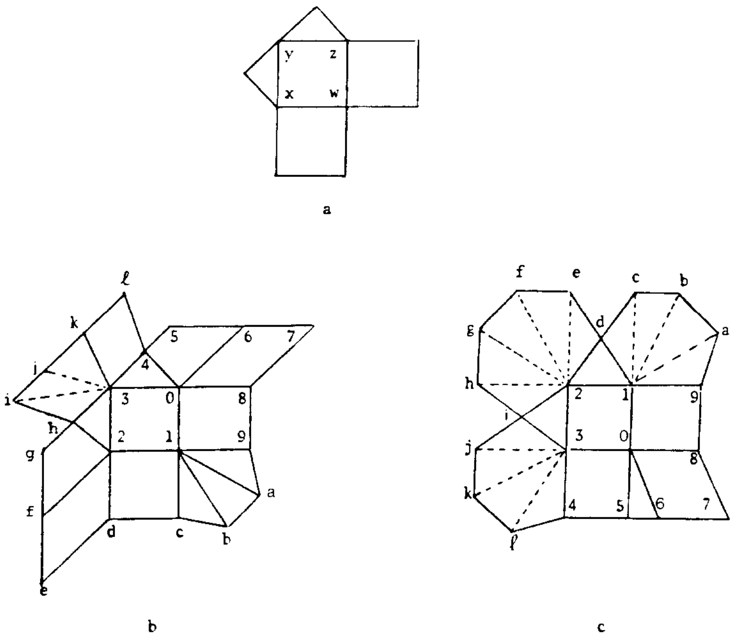


FIGURE 4

A_{456}, and if $b \neq 5$ then either ab or bc is the edge 46, while 46 is a diagonal. Therefore $d \neq 7$, and similarly $a \neq 7$.

$d = 6$ is also impossible, as otherwise $6 \in \{i, j, k\}$ and we obtain the contradiction C_6^+.

If $d = 4$, then 4 does not appear again, as it is already surrounded by the correct number of vertices. D_{46} implies $6 \notin \{c, e, f\}$, hence $6 \in \{a, b\}$ and $6 \in \{g, h\}$. If $h = 6$ we get the contradiction C_6^+, and if $g = 6$ then also $j = 6$ and again we get C_6^+.

The only possibility left for d is $d = 5$. Then $c \neq 7$ (otherwise $\{a, b\} = \{4, 6\}$, while D_{46}). We already know that $a \neq 7$, hence $b = 7$, $c \neq 6$ (otherwise A_{567}), hence $c = 4$, $a = 6$. Now 4 cannot appear anymore on the boundary of the configuration, and we get the contradiction C_4^-. (Or, stated differently, $l = 7$ and either 8 or 9 does not see 4.)

We conclude that $\deg w = 6$ is impossible. Hence $\deg w = 5$. This leads to the configuration of Fig. 4c (where $wxyz = 0321$), which again depicts the entire map M. As before, all the facets here are distinct.

Clearly $d \in \{4, 5, 6, 7\}$. If $d \in \{4, 5\}$ then the only possibility to avoid the contradiction C_d^- is $d = 4$, $1c$, $2e$, $3l$ are diagonals; $1b$, $2f$, and $3k$ are edges. In this case also $\{c, e\} = \{5, l\}$; $e = 5$ would imply $a = 5$ and the contradiction C_5^-. Hence $c = 5$, which implies $b = 7$, $a = 6$ (otherwise A_{567}). Now

$e = l \notin \{5, 7\}$, hence $e = l \in \{6, 8, 9\}$, but $e \in \{8, 9\}$ yields the contradiction C_e^- and $e = l = 6$ implies $\{k, f\} = \{5, 9\}$ (consider the link of 5) leading to the contradiction D_{45}. Thus $d \notin \{4, 5\}$ and hence $d \in \{6, 7\}$.

Assume $d = 6$. Then $6 \in \{j, k, l\}$ and, as $\deg 6 \leqslant 6$, 36 must be a diagonal. In order to avoid C_6^+ or C_6^- one of $1c$, $2e$ must be an edge and the other a diagonal. If $1c$ is a diagonal then $1b$ is an edge, $b = 4$ (otherwise D_{56} or D_{67}), $a = 5$ (otherwise A_{456}), and $c = 7$. Since 3 already sees 5 and the edge 56 must appear twice on the boundary, we get $e = 5$ and hence the contradiction C_5^-. Therefore $1c$ is an edge, $2e$ is a diagonal, and $2f$ is an edge. 3 already sees 5, hence $E_{56} \Rightarrow 5 \in \{c, e\}$. $e = 5$ would imply $5 \in \{a, b\}$ and the contradiction C_5^-, hence $c = 5$, which, in turn, implies $b = 7$ (otherwise A_{456}) and $a = 4$.

Now, in order to avoid the contradiction C_5^-, $1b$ $(=17)$ must be a diagonal, which leads to the contradiction D_{45}. Hence $d \neq 6$.

The only possibility left is $d = 7$. Clearly $i \neq 7$ (as 2 already sees 7) and $b \neq 5$ (otherwise $\{a, c\} = \{4, 6\}$ and A_{456}). Assume $b = 6$. Then $1b = 16$ is an edge (otherwise D_{45}) and similarly $1c$ is an edge. As there are already five triangles and 2 belongs to two of them (which means $\deg 2 = 6$) the sixth triangle must be incident to 2 too, hence $3k$ is an edge and $3j$, $3l$, $1a$ are diagonals. Now $\{i, j, k, l\} = \{6, 7, 8, 9\}$, D_{69} and D_{68} imply $l = 6$, $\{i, j\} = \{8, 9\}$, hence $k = 7$. This implies $j = 9$, $i = 8$ (otherwise A_{789}), but now 78 is both an edge and a diagonal, a contradiction.

Thus necessarily, $b = 4$. As before, $1b = 14$ must be an edge. $\{a, c\} = \{5, 6\}$ implies $l \neq 6$ (otherwise A_{546}). Since $p_3 = 6$ and $\deg 2 = 6$, at most one of $1a$, $1c$ is an edge. If one of them is an edge then $3k$, too, is an edge and $3j$, $3l$ are diagonals, while if both $1a$ and $1c$ are diagonals, then precisely two of $3j$, $3k$, $3l$ are edges. We shall examine each of these possibilities and see that each of them leads to a contradiction.

If $1c$ is an edge, then D_{49}, D_{4k} imply $9 \in \{i, j\}$. If $i = 9$ then $D_{68} \Rightarrow \{j, l\} = \{6, 8\} \Rightarrow l = 8, j = 6, k = 7 \Rightarrow A_{678}$. Hence $j = 9$, which together with D_{68} implies $l = 8$, $i = 6$, $k = 7$, but now 67 is both an edge and a diagonal.

If $1a$ is an edge then D_{4k}, $D_{47} \Rightarrow j = 7$; $D_{68} \Rightarrow \{i, l\} = \{6, 8\} \Rightarrow l = 8$, $i = 6$, $k = 9$. Now $D_{69} \Rightarrow a = 5$, $c = 6$ and the edge 67 appears three times!

Finally, if both $1a$ and $1c$ are diagonals, then D_{49}, $D_{47} \Rightarrow l \notin \{7, 9\} \Rightarrow l = 8$. Now $7 \in \{j, k\}$ and $k \neq 6$. If $k = 7$ then $j = 9$, $i = 6$ (otherwise A_{678}), but now D_{47}, E_{67}, and E_{89} imply that all the three of $3j$, $3k$, $3l$ must be edges–which as mentioned earlier is not true. (It would cause $p_3 > 6$.) If, on the other hand, $k = 9$, then $j = 7$, $i = 6$, then $a = 6$ (otherwise the edge 67 appears three times), and E_{69}, E_{78}, D_{49} lead to the same contradiction, namely $p_3 > 6$. Thus we conclude that Case 5 has no topological realization.

4. Two Realizable Cases

We turn now to the two remaining cases of Table I. We shall see that Case 1 has six distinct topological realizations—the maps $\mathscr{A}$,..., $\mathscr{F}$ of Fig. 1—and Case 2 has two distinct topological realizations—the maps $\mathscr{G}$, $\mathscr{H}$ of Fig. 2.

Case 1. Here $V = 9$, $p = (16, 3, 0, \ldots)$. Let M be such a w.n.p. map. Each of the three quadrangles in M must have at least a common vertex with each of the other two quandrangles, as otherwise there would be a quadrangle in M which touches at least 17 distinct triangles, while $p_3 = 16$. Thus the configuration of the three quadrangles in M must be of one of the eight types A, B,..., H depicted in Fig. 5. (Here, vertices which are not identified explicitly are distinct.) Next we examine each of these types and try to reconstruct from it the entire map M. We shall see that Types A, H cannot be extended, while each of the six other types gives rise to a unique map satisfying our requirements.

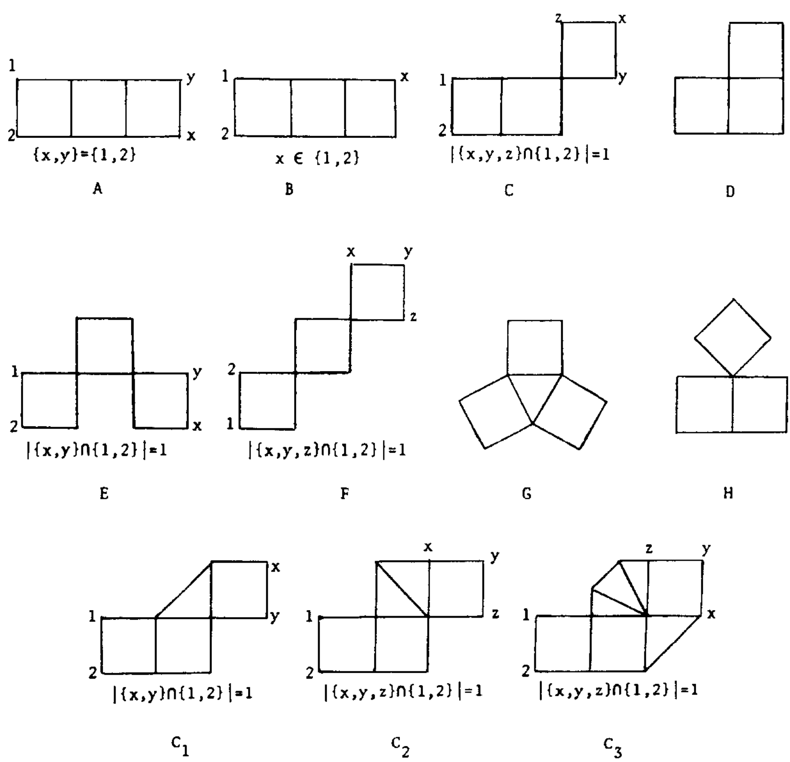


Figure 5

Type A. The configuration of Type A must be surrounded by distinct triangles as shown in Fig. 6a, and each of the bold vertices belongs to $\{7, 8, 9\}$. Now there exist just three edges which use just those three vertices, which means six edges on the boundary of Fig. 6a, but actually we see ten such edges on the boundary. Hence the configuration of Type A cannot be a part of M.

Type B. We check Type B and we shall see that if $x = 1$ it cannot be extended to the required map M, while if $x = 2$ it gives rise to a unique map M.

Assume $x = 1$. Then the map in Fig. 6b is a part of M. All the facets are distinct, as they touch the middle quadrangle and two triangles of M are missing. The triangle $46h$ touches all three quadrangles, hence h is not in any of the quadrangles, and we may take $h = 9$. Now, considering the neighborhoods of 4 and 6 we see that $\{f, g\} = \{7, 8\}$ and $\{i, j\} = \{2, 8\}$. As $i \neq g$ we get $8 \in \{f, j\}$ and, because of the symmetry of the con-

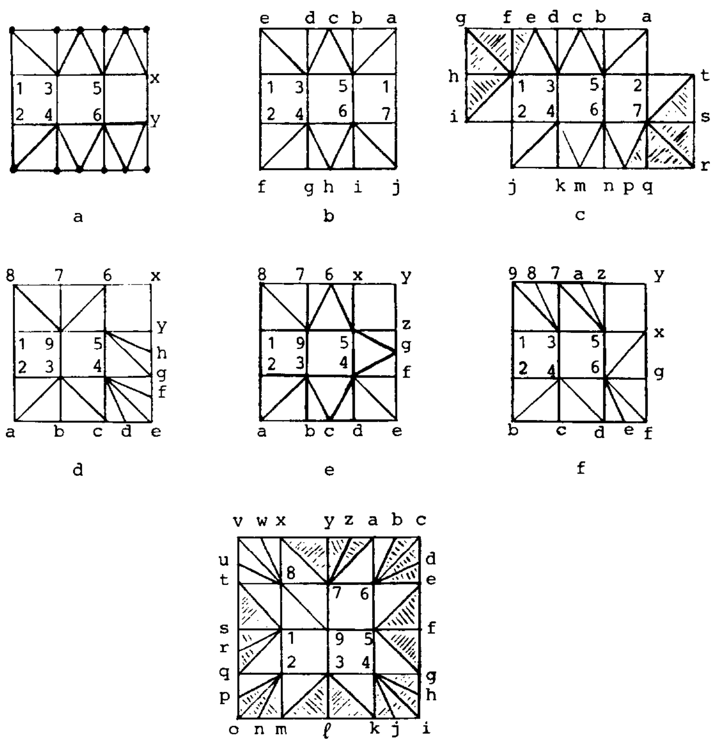


FIGURE 6

figuration, we may take $f=8$, which implies $g=7$. Considering the neighborhood of 3 we see that $\{c, d, e\}=\{7, 8, 9\}$, hence $d \neq 7$ (otherwise A_{978}). Similarly $\{i, j\}=\{2, 8\}$, $\{a, b, c\}=\{2, 8, 9\}$, and $b \neq i$. It follows that $c, d \in \{8, 9\}$ and, as $c \neq d$, we get $\{c, d\}=\{8, 9\}$. This implies $b \notin \{8, 9\}$ that is, $b=2$. Now $b=2 \Rightarrow n \neq 2$ (otherwise $A_{928}) \Rightarrow i=8$, $j=2$. Now 28 belongs to three triangles, namely, $2f4=284$, $ji6=286$, and $285=$ either $ba5$ or $bc5$. Thus $x \neq 1$.

Now assume $x=2$. Then Fig. 6c is a part of M, with not all the triangles distinct, and as in the previous case we may take $c=9$. Now $\{d, e\}=\{7, 8\}$ and $\{a, b\}=\{1, 8\}$, hence $b \neq d \Rightarrow 8 \in \{a, e\}$. Similarly $\{j, k, m\}=\{7, 8, 9\}$ and $\{m, n, p\}=\{9, 1, 8\}$ imply $m \in \{8, 9\}$. Considering the neighborhood of 2 we see that $\{i, j\}=\{a, t\}=\{8, 9\}$ which, together with $\{a, b\}=\{1, 8\}$, implies $a=8$, $b=1$, $t=9$. Now $\{j, m\}=\{8, 9\}$ implies $k \neq 7$, and either jk or km is 79. As the edge 79 already appears in two distinct triangles we get $d \neq 7$ and hence $d=8$, $e=7$, and for the same reason also $s=4$. The fact that 47 belongs to the triangles $47m$, $47j$, and $\{j, m\}=\{8, 9\}$ implies $r=8$. Similarly $B_{78} \Rightarrow q=3$, $B_{37} \Rightarrow p=1$, $B_{17} \Rightarrow f=6$, $B_{16} \Rightarrow g=n$. It follows that $g, i, j, m, n \in \{8, 9\}$, in particular, as $g \neq i$, we get $\{g, i\}=\{8, 9\}$, hence, considering the neighborhood of 1, we see that $h=5$. Now $i \neq 8$, as otherwise we get C_a, and this leads to $i=m=9$, $g=j=n=8$. Deleting the triangles which appear more than once and are therefore superfluous (shaded in Fig. 6c), we get precisely the map $\mathscr{A}$ of Fig. 1, which is indeed a w.n.p. map with nine vertices realizing $p=(16, 3, 0, \ldots)$. Thus Type **B** gives rise to a unique w.n.p. map.

Type C. Type C splits into the three types C_1, C_2, C_3 of Fig.. 5.

Type C_1. The map in Fig. 6d is a part of M. All the facets are distinct, as they all touch the middle quadrangle, and one triangle of M is missing here. Now $\{a, b, c\}=\{6, 7, 8\} \Rightarrow b \neq 7$ (otherwise A_{678}). $x \neq 7 \neq y$ as 6 already sees 7, therefore $|\{1, 2\} \cap \{x, y\}|=1 \Rightarrow 8 \in \{x, y\}$. $y \neq 8$ (as $b \neq 7 \Rightarrow E_{68}$), hence $x=8$, which implies $b \neq 6$ (otherwise A_{768}) and therefore $b=8$.

If $y=1$ then $\{g, h\}=\{2, 7\}$. Since $h \neq 2$ (otherwise A_{218}) we have $g=2$, $h=7$; $c=6$ would imply $d \neq 6$, $d \neq 8$, $d \neq 1$ (D_{16}), $d \neq 7$ (A_{768}), which is impossible, hence $c=7$, $a=6$. As $d \neq 6$ (otherwise A_{678}) and clearly $7 \neq d \neq 8$, we get $d=1$ which implies $\{e, f\}=\{6, 8\}$, but this is impossible as 68 already occurred twice.

Thus $y=2$. Now $D_{26} \Rightarrow a \neq 6 \Rightarrow a=7$, $c=6$. As $d \neq 2$ (D_{26}), $d \neq 8$ and $d \neq 7$ (otherwise A_{768}) we get $d=1$, therefore $g \neq 1$ and this, with $\{g, h\}=\{1, 7\}$, implies $g=7$, $h=1$. Now $B_{12} \Rightarrow f=2$, $e=8$. As each of the edges 16, 17, 67 occurs in just one triangle, the missing triangle must be 167. Thus we obtained precisely the map $\mathscr{B}$ of Fig. 1 which is indeed a w.n.p. map satisfying all our requirements.

Type C_2. We show that this type is impossible. If it is possible, it yields the configuration depicted in Fig. 6e, where again all the facets are distinct and one triangle of M is missing.

$\{a, b, c\} = \{6, 7, 8\} \Rightarrow b \neq 7$ (otherwise A_{678}), hence $\{x, y\} \neq \{7, 8\}$ (otherwise $x = 8$, $y = 7$ which yield $b \neq 6$, $b \neq 8$ (A_{768} or A_{687})); $|\{x, y, z\} \cap \{1, 2\}| = 1$ and $\{x, y, z, g\} = \{1, 2, 7, 8\}$ implies $\{7, 8\} \subset \{x, y, z\}$ and $g \in \{1, 2\}$. Since $E_{78} \Rightarrow \{x, z\} \neq \{7, 8\}$, we get $\{y, z\} = \{7, 8\}$, hence $\{x, g\} = \{1, 2\}$. Now $B_{78} \Rightarrow \{a, b\} \neq \{7, 8\} \neq \{b, c\} \Rightarrow b = 6$. If $a = 7$ then $c = 8$, $d \neq 6$, and also $B_{78} \Rightarrow d \neq 7$, hence $\{e, f\} = \{6, 7\}$, but this is impossible as 67 already occurred twice. Thus $a = 8$, $c = 7$, $d \neq 6$, $d \neq 8$ (otherwise A_{678}), hence $\{e, f\} = \{6, 8\}$, $\{d, g\} = \{1, 2\}$. If $x = 1$ then $g = 2$, $d = 1$, but now 17, 18 are edges while $1z$ is a diagonal and $z \in \{7, 8\}$. Hence $x = 2$. This implies $g = 1$, $d = 2$. As $z \neq 8$ (otherwise A_{187}) we have $y = 8$, $z = 7$, but now 27 is both an edge and a diaonal. Thus Type C_2 is not possible.

Type C_3. $\{b, c, d\} = \{7, 8, 9\}, c \neq 8$ (otherwise A_{789}). $|\{1, 2\} \cap \{x, y, z\}| = 1$, $\{x, y, z, a\} = \{1, 2, 8, 9\}$ thus $a \in \{1, 2\}$. 89 cannot be a diagonal, thus either $\{8, 9\} = \{x, y\}$ or $\{8, 9\} = \{y, z\}$. Thus $c \neq 9$ (otherwise the edge 89 occurs three times), hence $c = 7$ and $\{b, d\} = \{8, 9\}$.

First case: $\{8, 9\} = \{x, y\}$. Then $z \in \{1, 2\}$. If $z = 1$ then $x \neq 9$ (D_{19}), thus $y = 9$, $x = 8$ leading to the contradiction A_{198}. Therefore $z = 2$. Now $x \neq d$ and $\{x, y\} = \{b, d\}$ implies $x = b$, but now $2b = 2x$ is both an edge and a diagonal. Thus the first case is not possible.

Second case: $\{8, 9\} = \{y, z\}$. Then $x \in \{1, 2\}$. If $x = 1$, then $y = 8$, $z = 9$ (otherwise A_{198}) but now 19 would be an edge and a diagonal, thus $x = 2$. Hence $a = 1$, $z = 8$, $y = 9$ (otherwise A_{198}), $b = 9$ (D_{28}). This configuration is also of type C_1, which already has been considered.

Type D. Type D gives rise to the map in Fig. 6g, where of course not all triangles are distinct. Here $\{e, f, g\} = \{1, 2, 8\}$ and $f \neq 1$ (otherwise C_1 or A_{218}).

Assume $f = 2$. Then $1 \in \{e, g\} \Rightarrow q = 5$ $(B_{12}) \Rightarrow p = 8$ $(B_{25}) \Rightarrow r \in \{4, 6\}$ (B_{15}). $(g = 1 \Rightarrow r = 4$, $e = 1 \Rightarrow r = 6.)$ $E_{78} \Rightarrow t \neq 7 \Rightarrow s = 7$. $6 \in \{r, t\} \Rightarrow a = 1$ $(B_{67}) \Rightarrow e \neq 1 \Rightarrow e = 8 \Rightarrow g = 1 \Rightarrow r = 4$, $t = 6 \Rightarrow b = 8$ (B_{16}) which is impossible since $e = 8$. Hence $f \neq 2$. Thus $f = 8$. $\{e, g\} = \{1, 2\} \Rightarrow t = 5$ (B_{18}), $u = 2$ (B_{58}). Assume $e = 1$. Then $g = 2$, $s = 6$ $(B_{15}) \Rightarrow r = d \neq 7$ (B_{16} and 7 already sees 6) and this, together with $\{q, r\} = \{4, 7\}$ implies $d = r = 4$, $q = 7$. Now $q = 7 \Rightarrow m \neq 7$, $\{k, l, m\} = \{6, 7, 8\} \Rightarrow l \neq 7$ (otherwise C_7) $\Rightarrow k = 7 \Rightarrow j = 1$ $(B_{47}) \Rightarrow i = 6$ $(B_{14}) \Rightarrow h = 8$ (as 4 must see 8), but this is not possible, as $f = 8$.

Thus $e = 2$, $g = 1$. $B_{15} \Rightarrow s = 4$, $B_{14} \Rightarrow r = h$. Clearly $\{q, r\} = \{6, 7\}$, hence $a = 1$ (B_{67}). Now if $r = 7$ then $q = 6$, and $B_{16} \Rightarrow b = 2$ which is impossible, as $e = 2$. Hence $r = h = 6$, $q = 7$. Now $q = 7 \Rightarrow m \neq 7$ and $\{k, l, m\} =$

$\{6, 7, 8\} \Rightarrow l \neq 7$ (otherwise C_7) $\Rightarrow k = 7 \Rightarrow \{i, j\} = \{2, 8\} \Rightarrow v = 4$ (B_{28}). $j = 8$ would imply $w = 7$ (B_{47}) which is impossible as 8 already sees 7, hence $j = 2$, $i = 8$, $B_{48} \Rightarrow w = 6 \Rightarrow x = 3$, $B_{27} \Rightarrow p = 4$, $B_{37} \Rightarrow y = 4$, $l = 8 \Rightarrow m = 6$, $z = 2$. $B_{16} \Rightarrow b = 4$, $B_{46} \Rightarrow c = 8$, $B_{68} \Rightarrow d = 3$, $B_{24} \Rightarrow o = 8$, $n = 5$. Now omitting the double (or even triple) triangles (shaded in Fig. 6g) we get precisely the map $\mathscr{C}$ of Fig. 1, which is indeed a w.n.p. map satisfying all our requirements.

Type E. We consider first the possibility that the unique vertex in $\{x, y\} \cap \{1, 2\}$ is 1. Assume $y = 1$. Then x is not identical to any of the other vertices in the configuration, and we may take the labeling of the vertices as in Fig. 7a with $x = 9$. Thus the three quadrangles contain all the nine vertices of the map. Let $47z$ be the triangle containing the edge 47. Then z must be in one of the two side quadrangles, and because of the symmetry we may take $z \in \{2, 3\}$. Now $D_{24} \Rightarrow z = 3$, and we get the configuration of Fig. 7b, with $x = 9$, $y = 1$. Now $\{a, b\} = \{8, 9\}$, and $D_{18} \Rightarrow a = 9$, $b = 8$, but this yields the contradiction A_{198}. Hence $y \neq 1$.

Assume now $x = 1$. Then $y = 9$, and the vertex z of the triangle $47z$ is either 3 or 8. If $z = 3$ we are back to Fig. 7b, this time with $x = 1$, $y = 9$, but now $\{a, b\} = \{8, 9\}$ which is not possible, as 89 is a diagonal. Thus $z = 8$ and we are led to Fig. 7c. (Note that the third vertex in the triangle $48z$, $z \neq 7$, must be 3). Of course, not all the triangles here are distinct.

Clearly $n = 9$, $\{e, f\} = \{2, 3\}$, and $B_{19} \Rightarrow a = 4$. Assume $e = 3$, $f = 2$. As $\{b, c, d\} = \{2, 5, 6\}$ and D_{24}, D_{46}, $d \neq 2$, we see that $b = 5$, $c = 2, d = 6$. Now $B_{26} \Rightarrow g = 9$, $B_{69} \Rightarrow h = 3$, $B_{59} \Rightarrow m = 2$. Now $\{i, j\} = \{1, 8\}$, $\{j, k, l\} = \{1, 3, 8\}$, and $i \neq k$ implies $k = 3$, $1 \in \{j, l\}$ which is impossible, as 13 is a diagonal. Thus $e = 2, f = 3$.

Now $B_{34}, D_{46} \Rightarrow b = 5$, $\{c, d\} = \{3, 6\}$. $B_{36} \Rightarrow g = 9 \Rightarrow j \neq 9$ hence $B_{56} \Rightarrow c \neq 6 \Rightarrow d = 6$, $c = 3$. Now $B_{69} \Rightarrow h = 2$, $B_{59} \Rightarrow m = 3$, $\{i, j\} = \{1, 8\}$, $\{j, k, l\} = \{1, 2, 8\}$, D_{13}, and $i \neq k$ imply $k = 2$, $l = 8, j = 1$, $i = 8$. Omitting the double triangles (shaded in Fig. 7c), we get precisely the map $\mathscr{D}$ of Fig. 1 which indeed satisfies all our requirements.

Next we have to consider the possibility that the unique vertex in $\{x, y\} \cap \{1, 2\}$ is 2 (Fig. 7a). The case in which $x = 9$, $y = 2$ is identical, up to the labeling of the vertices, to the case $x = 1$, $y = 9$, already dealt with. Thus it remains to check the possibility $x = 2$, $y = 9$. Consider again the triangle $47z$. Clearly $z \neq 2$, as otherwise 24 is both an edge and a diagonal. Because of the symmetry it suffices to consider the case $z = 3$. This brings us back to Fig. 7b, this time with $x = 2$, $y = 9$. But now, as before, $\{a, b\} = \{8, 9\}$ and 89 is both an edge and a diagonal—a contradiction. Hence there is just one possibility to extend Type E to the required w.n.p. map.

Type F. The only interesting possibility here is $y = 1$, since all the

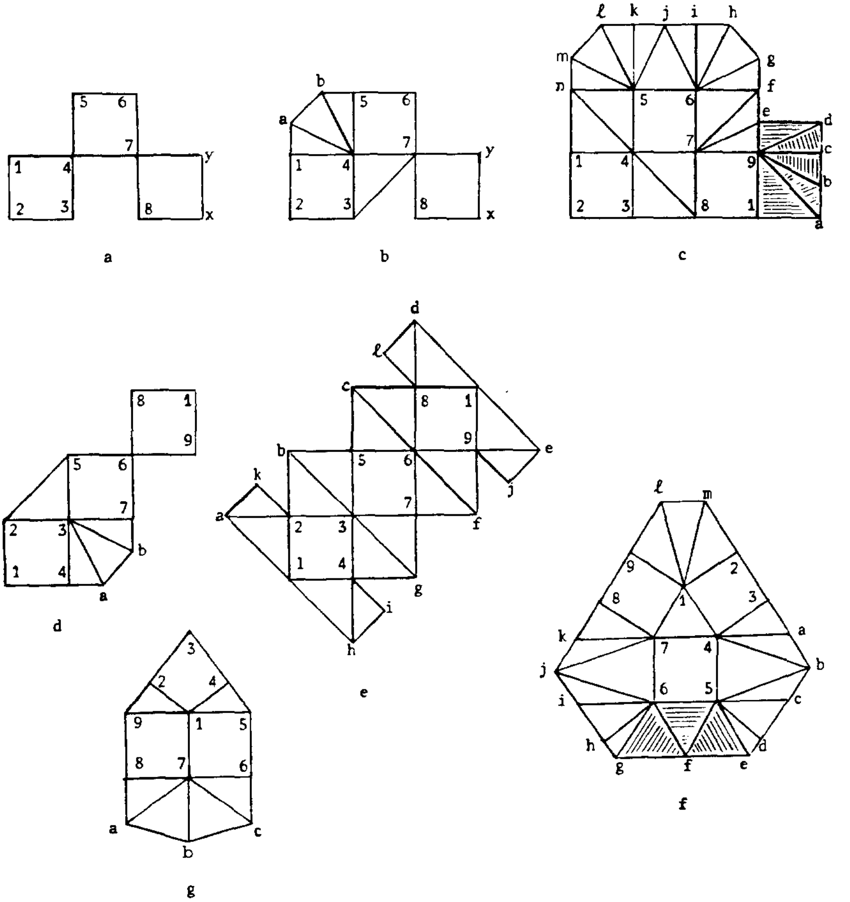


FIGURE 7

other possibilities are already included in Type E. Thus we take the labeling of the vertices of the three quadrangles to be as in Fig. 7d. Now 235 cannot be a triangle in the map M, as otherwise (see Fig. 7d) $\{a, b\} = \{8, 9\}$ so that 89 is both an edge and a diagonal. Because of the symmetry, also none of 568, 812, 437, 769, 914 is a facet in the map M. Thus we are led to the configuration depicted in Fig. 7e.

A priori, the facets in Fig. 7e are not necessarily distinct. However, considering the neighborhood of 1 we see that $\{a, h\} = \{d, e\} = \{5, 7\}$. Thus all the facets here touch the "middle" quadrangle, and they are therefore distinct. Since we have here the right number of facets of each type, it follows that Fig. 7e depicts the entire map M.

Now, in addition to $\{a, h\} = \{d, e\} = \{5, 7\}$ we have also $\{b, g\} = \{8, 9\}$,

$\{c, f\} = \{2, 4\}$. Six triangles must meet at each vertex which belongs to just one quadrangle. Thus $2 \in \{j, l\}$, as clearly $k \neq 2$ and $i \neq 2$ (D_{24}). Similarly also $4 \in \{j, l\}$ and $\{i, k\} = \{8, 9\}$.

Assume $a = 5$. Then $h = 7$, $B_{25} \Rightarrow c \neq 2 \Rightarrow c = 4, f = 2 \Rightarrow j \neq 2 \Rightarrow j = 4$, $l = 2$. Considering the neighborhood of 2 we see that $\{b, k, d\} = \{7, 8, 9\}$, hence $d = 7$. Now $d = 7 \Rightarrow e = 5 \Rightarrow k \neq 9$ (B_{59}) $\Rightarrow k = 8$, $i = 9$, $g = 8$, $b = 9$ and the edge 59 appears three time—a contradiction.

Thus $a = 7$, which implies $h = 5$. $B_{45} \Rightarrow c \neq 4 \Rightarrow c = 2, f = 4 \Rightarrow j \neq 4 \Rightarrow l = 4$, $j = 2$. Considering the neighborhood of 2 we see that $\{b, k, e\} = \{5, 8, 9\}$, which implies $e = 5$. Now $e = 5 \Rightarrow d = 7 \Rightarrow k \neq 8$ (B_{78}) $\Rightarrow k = 9$, $b = 8$, $g = 9$, $i = 8$. This is indeed the map $\mathscr{E}$ of Fig. 1 (drawn differently), and is indeed a w.n.p. map satisfying our requirements.

Type G. We label the vertices of the quadrangles as in Fig. 7f. We claim that the trianggle 147 is a facet of M. Indeed, let x be a vertex in the facet A containing 14 and different than 1234 ($1 \neq x \neq 4$). Then A is a triangle and x belongs to at least one of the other two quadrangles. Thus each of $1x$. $4x$ is either an edge or a diagonal in one of the quadrangles, and an edge in A. This is possible only if $x = 7$.

This gives rise to the configuration of Fig. 7f, which must be a part of M. Of course, not all the triangles here are distinct. Clearly $\{a, b\} = \{8, 9\}$, $\{j, k\} = \{2, 3\}$, and $\{l, m\} = \{5, 6\}$. As the edge 56 already exists in two facets, the last fact implies that the triangles $ml1$ and $56f$ coincide, that is, $f = 1$. Similarly $\{l, m\} = \{5, 6\}$ implies also that the triangles $1m2$, $1l9$ coincide with $5fe$, $6fg$, not necessarily in this order, hence $\{e, g\} = \{2, 9\}$. Now $\{b, c, d, e\} = \{2, 3, 8, 9\}$, B_{23} implies that 2, 3 are not adjacent here and $b \neq 2, 3$. Hence $e = 2$, $g = 9$, $c = 3$. Similarly (symmetry!) $i = 8$, $e = 2 \Rightarrow m = 5$, $l = 6$. If $k = 3$ then $j = 2$, $h = 3$, $8 \in \{a, b\}$, and the edge 38 appears three times. Hence $k = 2, j = 3$, $h = 2$, and similarly $a = d = 9$, $b = 8$. Omitting the double triangles (shaded) we get the map $\mathscr{F}$ of Fig. 1 which is indeed a w.n.p. map satisfying the requirements.

Type H. If Type H gives rise to a w.n.p. map M as required, then the configuration of Fig. 7g is a part of M, and here $\{a, b, c\} = \{2, 3, 4\}$. Now $b \neq 3$ as otherwise A_{234}. Thus one of the edges ab, bc is 24, but this is in contradiction to the fact that 24 is a diagonal. Hence Type H does not yield a w.n.p. map as required.

This ends the investigation of Case 1.

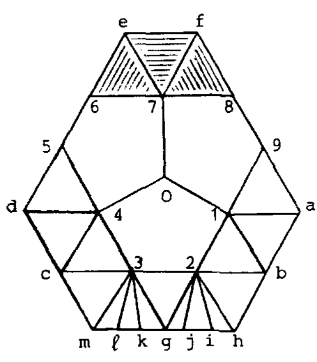
Case 2. Here $v = 10$, $p = (15, 0, 3, 0, \ldots)$. Assume there is such a map M. From (10) (with $j = 1$) it follows that each of the three pentagons in M shares an edge with each of the other two pentagons. We distinguish two cases both of which will be realizable.

Case a. The three pentagons share a common vertex. Then the configuration depicted in Fig. 8a is a part of M. Clearly, not all the triangles here are distinct. However, the 15 non-shaded triangles are distinct, as they all touch the pentagon 01234. Thus the map M actually consists of these 15 triangles and the three pentagons, and the three shaded triangles should be identified with three of the non-shaded triangles.

Considering the neighborhood of the vertex 7, we see that $\{e, f\} = \{2, 3\}$, which implies that the triangle $ef7$ is actually the triangle $23g$ (as the edge 23 belongs to just two facets), and therefore $g = 7$. Now $\{b, h, i, j\} = \{5, 6, 8, 9\}$ and D_{57}, D_{79} imply $j \in \{6, 8\}$. Note also that $\{a, b\} = \{5, 6\}$, $\{c, d\} = \{8, 9\}$.

Now, if $j = 6$ then $b \neq 6$, hence $b = 5$, $a = 6$, $\{i, h\} = \{8, 9\}$, and the edge 89 appears three times. Hence $j = 8$. This implies $f = 2$, $e = 3$. The 3-fold symmetry of the figure (with 0 as center) now implies $a = 6$, $b = 5$, $c = 9$, $d = 8$. As $\{h, i\} = \{6, 9\}$, B_{56} implies $h = 9$, $i = 6$. The remaining vertices are now determined by the 2-fold symmetry (with 07 as axis) to be $k = 6$, $l = 8$, $m = 5$, and it is easily checked that the map thus obtained is indeed a w.n.p. map with the required properties. It is the map $\mathscr{G}$ of Fig. 2.

Case b. The three pentagons do not share a common vertex. Then Fig. 8b depects the entire map, and all the facets here are distinct, as they all touch the middle pentagon. The fact that any two pentagons must have a common edge implies $6 \in \{b, c, d\}$ and $c \neq 9$. Clearly $a \neq 9$, hence $9 \in \{b, d\}$ and this implies $j = 9$ (otherwise, for 3 and 4 to see 9 we must have $k = 9$, $9 \in \{g, h\}$, and we get the contradiction C_9^+). Similarly (by symmetry) $a = f$. As $6 \in \{b, c, d\}$, 6 already sees 0, 1, 2, 4, and in order that 6 see 3 we must have $6 \in \{g, h, i\}$. Hence $c = 6$ would imply the contradiction C_6^-, and therefore $6 \in \{b, d\}$. Thus $\{a, c\} = \{5, 7\}$. Now $b = 6$ would imply A_{567}, hence $b = 9$ and $d = 6$. The symmetry of the configuration now implies $a = f = 7$, $c = 5$. For 2 and 4 to see 8 we must have $k = e = 8$. In



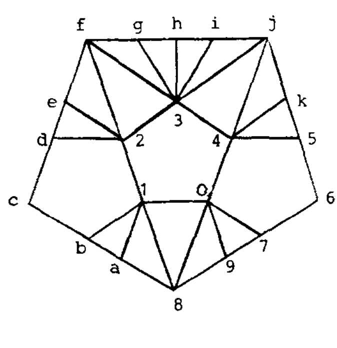


FIGURE 8

order that the edge 59 appear twice we must have $i=5$, and by symmetry $g=6$, thus $h=8$. Now it is easily checked that the map thus obtained is indeed a w.n.p. map with the required properties. It is the map $\mathscr{H}$ of Fig. 2.

Thus Case 2 has just two topological realizations. This completes the proof of our theorem.

5. Remarks

Alternative Construction

Let T_1 be the triangulated Möbius strip depicted in Fig. 9. Note that T_1 is weakly neighborly and has a pentagonal boundary which contains all the five vertices of T_1. If M is any w.n.p. map which contains a pentagonal facet F, then the removal of F yields a w.n.p. map with a pentagonal boundary and the glueing of T_1 to this map done by a proper identification of their boundaries yields a new map M_1. It is clear that M_1 is a w.n.p. map, has the same number of vertices as M, is not orientable and its genus is greater by $\frac{1}{2}$ than that of M. We say that M_1 is obtained from M by replacing the pentagonal facet F by the Möbius strip T_1.

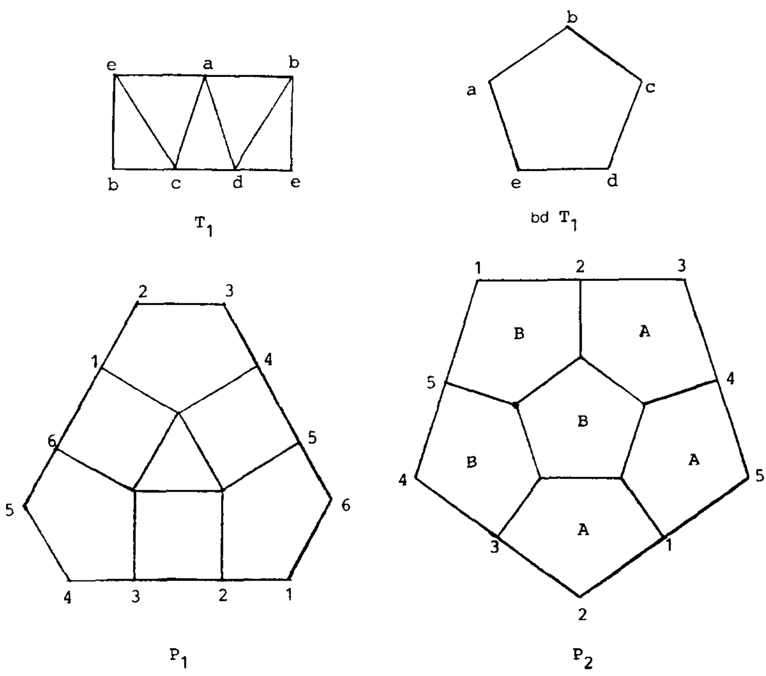


FIGURE 9

Now four of our eight w.n.p. maps can be obtained from w.n.p. maps of lower genus by application of this replacement operation. The map $\mathscr{D}$ is isomorphic to the map obtained in this manner from the map $\mathscr{B}$ in [AB2, Fig. 1], or also to the map obtained by a double application of this operation to the torus $\mathscr{E}$ in [BA2, Fig. 1]. The map $\mathscr{F}$ is isomorphic to the map obtained from the map $\mathscr{C}$ in [AB2, Fig. 1] by replacing the pentagonal facet by a Möbius strip, or also to the map obtained by a triple application of this operation to the projective plane P_1 ($g = \frac{1}{2}$) in Fig. 9.

Both maps $\mathscr{G}$, $\mathscr{H}$ are obtainable in this manner from the map $\mathscr{D}$ of [AB2, Fig. 1]. $\mathscr{G}$ by applying the operation at the facet 12659, and $\mathscr{H}$ by applying it at the facet 01987. Alternatively, both $\mathscr{G}$ and $\mathscr{H}$ are obtainable by a triple application of this operation to the projective plane P_2 ($g = \frac{1}{2}$) in Fig. 9: $\mathscr{G}$ by applying it to the three facets A, and $\mathscr{H}$ is obtained by applying it to the three facets B. Both w.n.p. maps P_1 and P_2 are described in detail in [B].

On the other hand, none of the maps $\mathscr{A}$, $\mathscr{B}$, $\mathscr{C}$, $\mathscr{E}$ is obtainable from a w.n.p. map of lower genus by means of this replacement operation, nor by another replacement (replacement by Császár's torus) described in [AB2, Sect. 3].

Symmetry Groups

Obviously, the symmetry group of each of our eight w.n.p. maps is a subgroup of the symmetry group of the fragment of the map consisting of the quadrangles (or pentagons) only. Thus it is easy to see that the symmetry group of $\mathscr{A}$ is Z_2 generated by (17) (36) (45); $\mathscr{B}$ is asymmetric; $\mathscr{C}$ and $\mathscr{D}$ have Z_2 as their symmetry group, generated by (17)(26)(35) and (17)(26)(35)(89), respectively; the symmetry group of $\mathscr{E}$ is $Z_2 \times Z_2$, with generators (28)(36)(49) and (24)(57)(89), and the symmetry group of $\mathscr{F}$ is S_3, generated by (147)(258)(369) and (29)(38)(47)(56).

In both maps $\mathscr{G}$ and $\mathscr{H}$ the symmetry group is S_3. In the case of $\mathscr{G}$ the generators are (147)(258)(369) and (14)(23)(59)(68) and in the case of $\mathscr{H}$ they are (025)(164)(739) and (01)(24)(56)(79). If we consider the method described above, obtaining $\mathscr{G}$ and $\mathscr{H}$ from the projective plane P_2 of Fig. 9 by a triple application of the Möbius strip operation, then in both cases the center of the symmetry S_3 is the bold vertex in P_2.

Construction Rule

The close relationship between the maps $\mathscr{F}$, $\mathscr{G}$ of Figs, 1, 2, respectively, suggests the following rule:

Let M be a w.n.p. map with $V+1$ vertices containing a 3-valent vertex w, let x. y. z be the three vertices adjacent to w, and let M' be the map obtained from M by deleting the vertex w and the edges incident to it, and adding the edges xy, yz, xz (or rather those of these edges which do not already exist in M) in the corresponding facets which contain w. Then M' is a w.n.p. map with V vertices and with the same genus and orientability as M.

Conversely, let M' be a w.n.p. map with V vertices containing a triangular facet Δ. If some (or all) of the three facets which share a common edge with Δ contain (together) each vertex not in Δ exactly once, then the map M obtained by adding a new vertex w inside Δ and the edges joining it to the vertices of Δ, and removing the proper edges of Δ, is a w.n.p. map with $V+1$ vertices and with the same genus and orientability as M.

References

[AB1] A. Altshuler and U. Brehm, Non-existence of weakly neighborly polyhedral maps on the orientable 2-manifold of genus 2, *J. Combin. Theory Ser. A* **42** (1986), 87–103.

[AB2] A. Altshuler and U. Brehm, The weakly neighborly polyhedral maps on the 2-manifold with Euler characteristic -1, *Discrete Comput. Geom.*, in press.

[B] U. Brehm, Weakly neighborly polyhedral maps on the projective plane, the Möbius strip, and the Klein bottle, to appear.

[BA1] U. Brehm and A. Altshuler, On weakly neighborly polyhedral maps of arbitrary genus, *Israel J. Math.* **53** (1986), 137–157.

[BA2] U. Brehm and A. Altshuler, The weakly neighborly polyhedral maps on the torus, *Geom. Dedicata* **18** (1985), 227–238.